Derivatives Pricing Under Bilateral Counterparty Risk

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Overview

- We consider the problem of valuing a diffusion-driven contingent claim under bilateral counterparty risk in a reduced-form setting similar to that of Duffie and Singleton [1999].
- The probabilistic valuation formulas derived under this framework cannot be used for practical pricing due to their recursive path dependencies. Instead, finite-difference methods are used to solve the quasi-linear partial differential equations (PDEs) that equivalently represent the claim value function.
- By imposing restrictions on the dynamics of the risk-free rate and the stochastic intensities of the counterparty default times, we develop path-independent probabilistic valuation formulas that have closed-form solution or can lead to computationally more efficient pricing schemes.

Overview

- Our risk-neutral valuation framework incorporates the so-called *wrong way risk* (WWR) as the two counterparty default intensities can depend on the derivatives value.
- The proposed modeling of WWR is more insightful than the widely-used incorporation of WWR in credit value adjustment (CVA) calculations via expected-discounted-loss type formulas as they do not represent the true *market price of counterparty credit risk*.
- Drawing upon the work of Ghamami and Goldberg [2014], we develop *calibration-implied formulas* that enable us to mathematically compare derivatives values in the presence and absence of WWR.
- We show that derivatives values under unilateral WWR need not be less than derivatives values in the absence of WWR. A sufficient condition under which this inequality holds is that the pre-default price process follows a semimartingale with independent increments.

The Model Setup

- Consider a fixed a probability space $(\Omega, \mathcal{F}, P)$ and a family $\{\mathcal{F}_t\}_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$ satisfying the usual conditions. Suppose that $X = (X_1, ..., X_n)^*$ is a Markovian state-variable vector process under an equivalent Martingale measure $\mathbb{Q}$.[2] The time-homogeneous diffusions X_i's have stochastic differentials of the form

$$dX_i(t) = \mu_i(t)dt + \sum_{j=1}^{d} a_{ij}(t)dW_j(t),$$

where $W_1, ..., W_d$ are independent 1-dimensional $\mathbb{Q}$ standard Brownian motions, and $\mu(t) \equiv \mu(X(t))$ and $a(t) \equiv a(X(t))$.

- Let the $n \times d$ matrix $a = \{a_{ij}\}$ denote the *dispersion matrix* of the above n-dimensional diffusion process. The symmetric *diffusion matrix* $b \equiv aa^*$ with elements $b_{ik} \equiv \sum_{j=1}^{d} a_{ij}a_{kj}$ is assumed to be non-negative-definite.

[2] $*$ denotes transpose.

The Model Setup

- Consider an arbitrage-free market model driven by the *n*-dimensional diffusion X. The money market account whose balance starts at one and then grows at some stochastic short interest rate $r_t \equiv r(X_t, t) \in \mathbb{R}$ is the numeraire.
- Consider a contingent claim maturing at the fixed date $T > 0$ with a single promised payoff given by the function $\Pi(x) : \mathbb{R}^n \mapsto \mathbb{R}$. The payoff function $\Pi(x)$ is not sign-definite.
- When the claim expires at T, it pays off $\Pi(X_T)$ at T, assuming the survival of both counterparties by time T.

Reduced-Form Modeling: Recovery of Market Value

- Suppose that the default time of counterparty i denoted by τ^i has a risk-neutral intensity process h^i such that

$$dH_t^i = (1 - H_t^i) h_t^i dt + dM_t^i,$$

where $H_t^i = 1\{\tau^i \leq t\}$ is the default indicator of counterparty i and M^i is a $\mathbb{Q}$ martingale; $i \in \{A, B\}$. We have assumed that $P(\tau^A = \tau^B) = 0$.

- Let U denote the value process of the claim from A's perspective. If a default occurs at time t, the time-t claim value is specified as $U_t^-(\gamma_t^A + \gamma_t^B)$ where

$$\gamma_t^A = 1\{t = \tau^A\} \left(1\{U_t^- < 0\}(1 - L_t^A) + 1\{U_t^- \geq 0\}\right),$$

and

$$\gamma_t^B = 1\{t = \tau^B\} \left(1\{U_t^- \geq 0\}(1 - L_t^B) + 1\{U_t^- < 0\}\right).$$

- Assume that the fractional loss processes L^i are bounded by 1 and are predictable. By U_t^- we mean the price of the claim just before default, i.e., $U_t^- \equiv \lim_{s \uparrow t} U_s$.

The Survival-Contingent Derivatives Value

- Set $\tau \equiv \tau^A \wedge \tau^B$.Consider the pre-default process V with $V_T = \Pi(X_T)$ and $V_t = U_t$ for $t < \tau$. That is, V_t is the market value of the contingent claim under bilateral default risk if there has been no default by time t.
- We assume that there exists a function $s^i(v,x,t) : \mathbb{R} \times \mathbb{R}^n \times [0,T] \mapsto \mathbb{R}^+$ such that $s^i_t \equiv s^i(V_t, X_t, t) = h^i_t L^i_t$ for $t \in [0,T]$ and $i = A, B$.
- Set $D_t \equiv \exp(\int_0^t r_u du)$. It can be shown that,

$$V_t = D_t E^{\mathbb{Q}}\left[\exp(-\int_t^T R_s ds)\frac{\Pi(X_T)}{D_T}\middle| X_t\right], \tag{1}$$

where $$R_t = s^A_t 1\{V_t < 0\} + s^B_t 1\{V_t \geq 0\}.$$

- The functional dependence of s^i on V captures WWR.

PDE for Survival-Contingent Claim Value

- Suppose that the short rate $r_t = r(X_t, t)$ for $t \in [0, T]$ for some function $r(x, t) : \mathbb{R}^n \times [0, T] \mapsto \mathbb{R}$ of the driving diffusion X.
- Based on the "Feynman-Kac" formula, equation (1) implies that V solves the backward Kolmogorov quasi-linear PDE:

$$\begin{aligned} \left(\mathcal{G}_x + \frac{\partial}{\partial t}\right) V(x,t) = \{ & r(x,t) \\ & + s^A(V(x,t),x,t)1[V(x,t) < 0] \\ & + s^B(V(x,t),x,t)1[V(x,t) \geq 0]\} V(x,t), \end{aligned} \tag{2}$$

for $x \in \mathbb{R}^n, t \in [0, T]$, where the infinitesimal generator of the X diffusion is:

$$\mathcal{G}_x \equiv \sum_{i=1}^{n} \mu_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} dX_i dX_j.$$

- The terminal condition restricting the survival-contingent claim value is:

$$V(x, T) = \Pi(x), \qquad x \in \mathbb{R}^n.$$

Short Rate Dynamics

- Suppose that the 1-dimensional positive process n is a $\mathcal{C}^2$ function of X_k for a given $1 \leq k \leq n$. We require that

$$r(x) = \frac{\mathcal{G}_{x_k} n(x)}{n(x)} + \lambda, \tag{3}$$

where λ is a constant and

$$\mathcal{G}_{x_k} n(x) = \mu_k \frac{\partial n}{\partial x_k} + \frac{1}{2} \frac{\partial^2 n}{\partial x_k^2} \sum_{j=1}^{d} a_{kj}^2 .$$

- It is not difficult to construct short rate processes based on (3). For instance, setting $n(X_k(t)) = \exp(-X_k^2(t))$ gives

$$r(X_k(t)) = -\mu_k(t) X_k(t) + \frac{1}{2} \left(X_k^2(t) - 1 \right) \sum_{j=1}^{d} a_{kj}^2(t) + \lambda .$$

Setting the drift μ_k equal to zero and choosing a positive λ, r becomes a diffusion that can stay positive almost surely.

The Auxiliary Probability Measure Change

- For any $0 \leq t \leq T$ define the process

$$N(t) = n(X_k(t)) \exp\left(- \int_0^t \frac{\mathcal{G}_{x_k} n(X_k(u))}{n(X_k(u))} du \right).$$

Set $n' \equiv \frac{\partial n(x)}{\partial x_k}$ and $n(t) \equiv n(X_k(t))$.

- Using Itô's formula and given the stochastic differential of $\log(N_t)$, we equivalently have,

$$N(t) = \exp\left\{ \sum_{j=1}^{d} \int_0^t \frac{n'(u)}{n(u)} a_{kj}(u) dW_j(u) - \frac{1}{2} \int_0^t \left(\frac{n'(u)}{n(u)} \right)^2 \sum_{j=1}^{d} a_{kj}^2(u) du \right\}.$$

That is, $\{N_t\}_{t \leq T}$ is a $\mathbb{Q}$-martingale assuming that the Novikov condition holds.

- Define the new auxiliary probability measure $\tilde{\mathbb{Q}}$ on $\mathcal{F}_T$ by

$$d\tilde{\mathbb{Q}} = N_T d\mathbb{Q} \quad \text{on} \quad \mathcal{F}_T.$$

- We know from Bayes' Theorem that

$$E^{\tilde{\mathbb{Q}}}\left[e^{-\int_t^T R_s ds} \frac{\Pi(X_T)}{e^{\lambda T} n_T} \middle| \mathcal{F}_t \right] = \frac{E^{\mathbb{Q}}\left[N_T e^{-\int_t^T R_s ds} \frac{\Pi(X_T)}{e^{\lambda T} n_T} \middle| \mathcal{F}_t \right]}{N_t} = \frac{V_t}{e^{\lambda t} n_t} \equiv \tilde{V}_t, \quad (4)$$

where by Girsanov Theorem, under $\tilde{\mathbb{Q}}$ the process X evolves with the drift change

$$dX_i(t) = \tilde{\mu}_i(t)dt + \sum_{j=1}^{d} a_{ij}(t) d\tilde{W}_j(t)$$

where $\tilde{W}_j$ are $\tilde{\mathbb{Q}}$ standard Brownian motions and for $i \neq k$, $\tilde{\mu}_i = \mu_i + \frac{n'}{n} \sum_{j=1}^d a_{ij} a_{kj}$ and $\tilde{\mu}_k = \mu_k + \frac{n'}{n} \sum_{j=1}^d a_{kj}^2$.

Default Intensity Dynamics

- Let $f(\tilde{V})$ be a $\mathcal{C}^2$ function and use Itô's formula to derive the stochastic differential of f. Given the PDE representation of the auxiliary value process $\tilde{V}$, setting the drift of the process $f(\tilde{V})$ equal to zero gives,

$$f'(\tilde{V}_t)\left(s_t^A 1\{\tilde{V}_t < 0\} + s_t^B 1\{\tilde{V}_t \geq 0\}\right)\tilde{V}_t + \frac{1}{2}f''(\tilde{V}_t)\beta = 0, \tag{5}$$

where $\beta \equiv (d\tilde{V}_t)^2 = (d\tilde{V}_t).(d\tilde{V}_t)$ is computed according to the Itô's multiplication rules that give:

$$\beta \equiv \sum_{i=1}^{n}\left(\frac{\partial \tilde{V}}{\partial x_i}\right)^2 \sum_{j=1}^{d} a_{ij}^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial \tilde{V}}{\partial x_i}\frac{\partial \tilde{V}}{\partial x_j}\sum_{k=1}^{d} a_{ik}(t)a_{jk}(t). \tag{6}$$

Set $\tilde{V}_x \equiv (\frac{\partial \tilde{V}}{\partial x_1}, \frac{\partial \tilde{V}}{\partial x_1}, ..., \frac{\partial \tilde{V}}{\partial x_n})^*$. Note that in matrix notation β takes the quadratic form $\beta = \tilde{V}_x^* b \tilde{V}_x$, where $b = aa^*$ is the non-negative-definite diffusion matrix. That is, β becomes non-negative due to b being non-negative-definite, and β will be positive when the diffusion matrix b is positive-definite.

Default Intensity Dynamics (Con'd)

- We suppose that each non-negative function s^i has the multiplicative form,

$$s^i(\tilde{v}, x, t) = g^i(\tilde{v})\,\beta(x, t), \qquad i = A, B, \tilde{v} \in \mathbb{R}, x \in \mathbb{R}^n, t \in [0, T], \tag{7}$$

where $g^i(\tilde{v}) : \mathbb{R} \mapsto \mathbb{R}^+$, and $\beta(x, t) : \mathbb{R}^n \times [0, T] \mapsto \mathbb{R}^+$.

- We require that the positive function $\beta(x, t)$ be given by (6), and so given the zero-drift condition (5) we can write,

$$-\frac{1}{2}\frac{f''(\tilde{v})}{f'(\tilde{v})} = g^B(\tilde{v})\tilde{v}^+ - g^A(\tilde{v})v^-, \qquad \tilde{v} \in \mathbb{R}.$$

Assuming that f solves the above linear ODE, we have,

$$f(\tilde{v}) = \int_0^{\tilde{v}} e^{-2\int_0^y [g^B(z)z^+ - g^A(z)z^-]dz} dy. \tag{8}$$

This function is increasing everywhere and hence invertible.

The Path-Independent Probabilistic Valuation Formula

- Assuming that $f(\tilde{V})$ is a $\tilde{\mathbb{Q}}$ martingale (not just a local martingale), we have,

$$f(\tilde{V}_t) = E^{\tilde{\mathbb{Q}}}[f(\tilde{V}_T)|X_t] = E^{\tilde{\mathbb{Q}}}\left[f\left(\frac{\Pi(X_T)}{e^{\lambda T} n_T}\right)\middle| X_t\right].$$

Given that f is invertible, we can write,

$$\tilde{V}_t = f^{-1}\left(E^{\tilde{\mathbb{Q}}}\left[f\left(\frac{\Pi(X_T)}{e^{\lambda T} n_T}\right)\middle| X_t\right]\right).$$

- Recall that $V_t = e^{\lambda t} n_t \tilde{V}_t$. Our path-independent probabilistic valuation formula becomes,

$$V_t = e^{\lambda t} n_t f^{-1}\left(E^{\tilde{\mathbb{Q}}}\left[f\left(\frac{\Pi(X_T)}{e^{\lambda T} n_T}\right)\middle| X_t\right]\right), \qquad t \in [0, T], \tag{9}$$

where f is specified in (8).

The Impact of Unilateral WWR on Derivatives Values

- Recall $V_t = E_t\left[e^{-\int_t^T (r_u + h_u^B)du}\Pi_T\right]$ under unilateral counterparty risk and zero recovery rate.
- Under WWR, h^B is defined as an increasing function of V, $V_t^W = E_t\left[e^{-\int_t^T (r_u + h_u^{B,w})du}V_T\right]$.
- When τ^B is independent of V and r, we have,

$$\begin{aligned} V_t^I &= P(\tau^B > T|\tau^B > t)E_t\left[e^{-\int_t^T r_u du}V_T\right] \\ &= E_t\left[e^{-\int_t^T h_u^{B,w}du}\right]E_t\left[e^{-\int_t^T r_u du}V_T\right], \end{aligned}$$

  where the right side above is the *calibration-implied formula*. It is not difficult to see V_0^W need not be lower than V_0^I.
- *Proposition 1: Suppose that V is a semimartingale with independent increments. Assume that the risk-free rate is constant. The covariance of two random variables $e^{-\int_0^T h(V_u)du}$ and V_T is non-positive and so $V_0^W \le V_0^I$.*

CVA and Wrong Way Risk

- Consider the unilateral case assume zero recovery rate. Suppose that $V_t \geq 0$ for all t from A's perspective. So, $V_0 = E[e^{-\int_0^T (h_u^B + r_u)du}\Pi(X_T)]$.
- Let $\hat{V}_0 = E[e^{-\int_0^T r_u du}\Pi(X_T)]$ denote the initial value of the claim in the absence of counterparty risk. the CVA expected-loss formula at time zero becomes, $\text{CVA} = E[e^{-\int_0^{\tau^B} r_u du}\hat{V}_{\tau^B} 1\{\tau^B \leq T\}]$.
- Note that,

$$\hat{V}_0 - V_0 = E\left[e^{-\int_0^T r_u du}\Pi(X_T)(1 - e^{-\int_0^T h_u^B du})\right],$$

so the equality $\hat{V}_0 - V_0 = \text{CVA} \equiv E\left[e^{-\int_0^{\tau} r_u du}\hat{V}_{\tau^B} 1\{\tau^B \leq T\}\right]$, need not hold when τ^B is not independent of $\hat{V}$ and r.

CVA_W need not exceed CVA_I

- Recall the CVA formula, $\text{CVA} = E[\tilde{D}_{\tau^B}\hat{V}_{\tau^B}1\{\tau^B \leq T\}]$, where assuming zero recovery rate, $\tilde{D}_t = \exp(-\int_0^t r_u du)$, and τ^B has the density density f.
- Assume that τ^B has a well-defined stochastic intensity h^B. Under WWR, h^B is an increasing function of $\hat{V}$. Under technical conditions, we have,

$$\text{CVA}_W = \int_0^T E\left[\tilde{D}_t\hat{V}_t h_t^{B,w} e^{-\int_0^t h_u^{B,w} du}\right] dt.$$

- The calibration-implied formula for CVA_I is derived as follows,

$$\text{CVA}_I = \int_0^T E[\tilde{D}_t\hat{V}_t] f_{\tau^B}(t) dt = \int_0^T E[\tilde{D}_t\hat{V}_t] E[h_t^{B,w} e^{-\int_0^t h_u^{B,w} du}] dt,$$

  the right side follows by noting that the model-calibration scheme is to approximate the model parameters by matching model-implied survival probabilities $E[e^{-\int_0^t h_u^{B,w} du}]$ to market-observed survival probabilities $P(\tau^B > t)$ for any $0 \leq t \leq T$ as closely as possible.

Summary and Future Work

- Our restrictions on the dynamics of the short rate and counterparty default intensities lead to path-independent probabilistic formulas for the survival-contingent claim value.
- Examples of short interest rate and counterparty default intensity specifications which satisfy these structural constraints should be further explored so that the foregoing framework can be tested empirically.
- The impact of WWR on derivatives values cannot be assessed in the absence of *calibration-implied formulas* that make the derivatives values in the presence and absence of WWR mathematically comparable. Deriving these formulas, we show that derivatives values under unilateral WWR need not be less than the derivatives value in the absence of WWR.
- We illustrate that the above-mentioned inequality holds when the pre-default price process follows a semimartingale with independent increments.

Appendix A: The Impact of Bilateral WWR on Derivatives Values

- *Proposition 2: The process* V_t

$$V_t = \hat{D}_t E_t \left[\hat{D}_T^{-1} V_T + \int_t^T \hat{D}_u^{-1} V_u \left(1\{V_u \geq 0\} h_u^A + 1\{V_u < 0\} h_u^B \right) du \right],$$

with $\hat{D}_t = \exp\left(\int_0^t (r_u + h_u^A + h_u^B) du \right)$ *for* $t \in [0, T]$*, and* $V_T \equiv \Pi_T$ *can be equivalently expressed as*

$$V_t = E_t \left[e^{-\int_t^T (r_u + h_u^B 1\{V_u \geq 0\} + h_u^A 1\{V_u < 0\}) du} V_T \right],$$

which is the survival-contingent risk-neutral derivatives price process assuming that both recovery rates are zero.

Bilateral WWR and Derivatives Values (Con'd)

- Suppose that both h^B and h^A are monotone increasing in V. Set $h^w \equiv h^{A,w} + h^{B,w}$. Using Proposition 2, we have,

$$\begin{aligned} V_0^{WW} &= E\left[e^{-\int_0^T (r_u + h_u^w) du} V_T\right] + \int_0^T E\left[V_u^+ h_u^{A,w} e^{-\int_0^u (r_s + h_s^w) ds}\right] du \\ &- \int_0^T E\left[V_u^- h_u^{B,w} e^{-\int_0^u (r_s + h_s^w) ds}\right] du, \end{aligned} \tag{10}$$

  where $V_t^+ \equiv V_t 1\{V_t \geq 0\}$ and $V_t^- \equiv -V_t 1\{V_t \geq 0\}$.

- The independent case gives,

$$\begin{aligned} V_0^I &= P(\tau > T) E\left[e^{-\int_0^T r_u du} V_T\right] \\ &+ \int_0^T E\left[e^{-\int_0^u r_s ds} V_u^+\right] P(\tau = \tau^A | \tau = u) f_\tau(u) du \\ &- \int_0^T E\left[e^{-\int_0^u r_s ds} V_u^-\right] P(\tau = \tau^B | \tau = u) f_\tau(u) du. \end{aligned}$$

The Calibration-Implied Formula

- Any calibration scheme is to ensure that model parameters are estimated such that the model-implied survival probabilities $E[e^{-\int_0^t h_u^w du}]$ match the market-implied survival probabilities $P(\tau > t)$ for any $t \in (0, T]$ as closely as possible. Similarly, it is to ensure that the model-implied default probabilities $E\left[\int_0^t h_u^i e^{-\int_0^u h_s ds} du\right]$ match the market-implied default probabilities $P(\tau = \tau^i, \tau \leq t)$, $i = A, B$ as closely as possible.
- So, the calibration-implied formula in the independent case can be derived as

$$\begin{aligned} V_0^I &= E\left[e^{-\int_0^T r_u du} V_T\right] E\left[e^{-\int_0^T h_u^w du}\right] \\ &\quad + \int_0^T E\left[e^{-\int_0^u r_s ds} V_u^+\right] E\left[h_u^{A,w} e^{-\int_0^u h_s^w ds}\right] du \\ &\quad - \int_0^T E\left[e^{-\int_0^u r_s ds} V_u^-\right] E\left[h_u^{B,w} e^{-\int_0^u h_s^w ds}\right] du. \end{aligned} \tag{11}$$

The derivatives values under bilateral WWR will now become mathematically comparable with the derivatives values in the independent case.

Appendix B: Valuation Under Right Way Risk

- Consider a contingent claim whose value is non-negative from A's perspective; we are interested in risk-neutral valuation under the default risk of merely counterparty B.
- Recall the proposed credit spread dynamics $s^B = g(\tilde{V})\beta$, with $\beta = \tilde{V}_x^* b \tilde{V}_x$ and the auxiliary value process $\tilde{V}_t = e^{-\lambda t} V_t / n_t$; λ being a constant and n being a 1-dimensional positive process which is a $\mathcal{C}^2$ function of X_k for a given $1 \leq k \leq n$.
- Suppose that the modeler wants the credit quality of counterparty B to be a decreasing function of the pre-default auxiliary value process and chooses $g(\tilde{V}) = c/\tilde{V}$, where c is a positive constant. This gives the convenient form $f(\tilde{v}) = \frac{1}{2c}(1 - e^{-2c\tilde{v}})$, and so,

$$V_t = -\frac{e^{\lambda t}}{2c} n_t \log E^{\tilde{\mathbb{Q}}} \left[\exp\left(-2ce^{-\lambda T} \frac{\Pi(X_T)}{n_T} \right) \middle| X_t \right],$$

where we know the $\tilde{\mathbb{Q}}$ dynamics of the diffusion X.

Appendix C: The Auxiliary Value Process and its PDE representation

- Viewing the auxiliary value process $\tilde{V}$ as a function of the n-dimensional X under $\tilde{\mathbb{Q}}$, and using multidimensional Itô's formula, we have

$$d\tilde{V}(t) = \frac{\partial \tilde{V}}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial \tilde{V}}{\partial x_i} dX_i(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \tilde{V}}{\partial x_i \partial x_j} dX_i(t) dX_j(t).$$

- Given (4), the PDE representation of the auxiliary value process $\tilde{V}$ via Feynman-Kac becomes

$$\left(\mathcal{G}_x + \frac{\partial}{\partial t} \right) \tilde{V}(x,t) =$$
$$\left(s^A(\tilde{V}(x,t), x, t) 1[\tilde{V}(x,t) < 0] + s^B(\tilde{V}(x,t), x, t) 1[\tilde{V}(x,t) \geq 0] \right) \tilde{V}(x,t)$$

with the terminal condition

$$\tilde{V}(x,T) = \frac{\Pi(x)}{e^{\lambda T} n(x)}, \qquad x \in \mathbb{R}^n.$$